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Multipole moments in general relativity

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Abstract. It is proved that a stationary space–time for which all the angular momentum multipole moments vanish is static and that a static space–time for which all the mass multipole moments vanish is flat.

1. Introduction

Multipole moments attract some interest in general relativity mainly because they may provide an algorithm for interpreting solutions. The idea is that, having obtained the multipole moments of a given solution of Einstein's equation, one may construct the Newtonian gravitational field with exactly the same moments and thus obtain the Newtonian analogue of the solution. A number of, generally inequivalent, definitions of multipole moments appear in the literature (Geroch 1970, Clarke and Sciama 1971, Hansen 1974, Hoenselaers 1976, Thorne 1977, and others).

We here study the Geroch–Hansen multipole moments. These multipole moments are defined only for stationary, asymptotically flat space–times. Precisely, one defines two collections of totally symmetric, trace-free tensors at spatial infinity, the mass multipole moments and the angular momentum multipole moments. But these multipole moments might be considered more as mathematical curiosities than actual physical notions. Although they seem reasonable and they obey some of the properties of the Newtonian multipole moments, very little is known about the physical information they carry. For instance, although it is expected that to virtually any collection of multipole moments there corresponds an essentially unique space–time, no proof is apparently known. Here, we present a result which indicates that the Geroch–Hansen multipole moments in fact carry information about the space–time. In addition, this result also constitutes the first step towards a uniqueness proof.

Theorem. Let (M, g_{ab}) be a stationary, vacuum, asymptotically flat at spatial infinity space–time. Then, if all the angular momentum multipole moments vanish, the space–time is static. Moreover, if, in a static space–time, all the mass multipole moments vanish, the space–time is flat.

If the space–time is vacuum only in some neighbourhood of infinity then the vanishing of the multipole moments implies that the space–time is static or flat only in that same neighbourhood of infinity. Of course, without any additional condition on the sources, one does not expect to be able to conclude anything about the non-vacuum region.

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The main tool we are using is the following.

Unique continuation theorem (Aronszajn 1957). Let M be a smooth, connected, m-dimensional manifold and let $(N, \{x^i\})$ be a chart of M. Let $\mathscr{D}$ be a linear, second-order, elliptic (i.e. the coefficients of the principal part of $\mathscr{D}$, $a^{ij}(x)\partial^2/\partial x^i \partial x^j$, form a positive definite matrix $\{a^{ij}(x)\}$ in N) differential operator with smooth (variable) coefficients. Let ϕ be a C^∞ scalar field on M which satisfies on N the differential inequality

$$|\mathscr{D}\phi(x)|^2 \leq B_N\left(\sum_{i=1}^{m} |\partial\phi(x)/\partial x^i|^2 + |\phi(x)|^2\right) \tag{1}$$

for some constant B_N. Assume that ϕ and all its derivatives vanish at some point in N. Then ϕ vanishes everywhere in N†.

2. The proof

We first recall the definition of multipole moments given by Hansen (1974).

Let (M, g_{ab}) be a stationary space–time which satisfies the vacuum Einstein equation and which is asymptotically flat at spatial infinity. So, in particular, we have the following.

(i) A time-like Killing vector ξ^a with 'norm' $\lambda = -\xi^a\xi_a$ and twist (potential) ω defined by $\nabla_a\omega = \epsilon_{abcd}\xi^b\nabla^c\xi^d$, where ∇_a is the derivative operator and ϵ_{abcd} an alternative tensor of (M, g_{ab}).

(ii) A three-dimensional manifold S, the manifold of trajectories of the Killing field ξ^a, with positive definite metric $h_{ab} = g_{ab} + \lambda^{-1}\xi_a\xi_b$.

(iii) A three-dimensional manifold $\tilde{S} = S \cup \Lambda$ with smooth, positive definite metric $\tilde{h}_{ab} = \Omega^2 h_{ab}$, where Λ is a single point (representing spatial infinity) and Ω is a smooth, positive scalar field on S. Moreover, we assume that $\Omega|_\Lambda = 0$, $\tilde{D}_a\Omega|_\Lambda = 0$, $\tilde{D}_a\tilde{D}_b\Omega|_\Lambda = 2\tilde{h}_{ab}|_\Lambda$ where $\tilde{D}_a$ is the derivative operator of $\tilde{h}_{ab}$. Now consider on $\tilde{S}$ the scalar fields

$$\tilde{\phi}_J = \Omega^{-1/2}\omega/2\lambda^{3/4} \qquad \tilde{\phi}_M = \Omega^{-1/2}\frac{\lambda^2+\omega^2-1}{4\lambda^{3/4}}.$$

It is a consequence of the Einstein equation that $\tilde{\phi}_J$ and $\tilde{\phi}_M$ are smooth on $\tilde{S}$ and that they satisfy the equations

$$\begin{aligned} \tilde{D}^m\tilde{D}_m\tilde{\phi}_J &= (\tfrac{1}{8}\tilde{R} + \tfrac{15}{8}\tilde{k}^4)\tilde{\phi}_J \\ \tilde{D}^m\tilde{D}_m\tilde{\phi}_M &= (\tfrac{1}{8}\tilde{R} + \tfrac{15}{8}\tilde{k}^4)\tilde{\phi}_M \end{aligned} \tag{2}$$

where $\tilde{R}$ is the scalar curvature of $\tilde{h}_{ab}$ and $\tilde{k}^4$ is a smooth scalar field on $\tilde{S}$ which depends only on $\tilde{\phi}_M$, $\tilde{\phi}_J$, and their first derivatives. Define, recursively, a set of tensor fields $P_{a_1\ldots a_s}$ on $\tilde{S}$ by $P = \tilde{\phi}_J$:

$$P_{a_1\ldots a_{s+1}} = \mathscr{C}[\tilde{D}_{a_1}P_{a_2\ldots a_{s+1}} - \tfrac{1}{2}s(2s-1)\tilde{R}_{a_1a_2}P_{a_3\ldots a_{s+1}}]$$

where $\mathscr{C}$ denotes the operation of taking the totally symmetric, trace-free part of a tensor and where $\tilde{R}_{ab}$ is the Ricci tensor of $\tilde{h}_{ab}$. *The value of $P_{a_1\ldots a_s}$ at the point Λ is the*

† The actual result is much stronger; e.g., one needs to assume a lower differentiability for the manifold M, the elliptic operator $\mathscr{D}$ and the scalar ϕ. In this case, instead of requiring the vanishing of ϕ and all its derivatives at some point of N, one assumes that ϕ has a zero of infinite order in the one-mean at some point of N. For details, see Aronszajn *et al* (1962).

2^s *angular momentum multipole moment of* (M, g_{ab}). One obtains the mass multipole moments similarly, starting from $\tilde{\phi}_M$. The multipole moments defined in this way depend on the particular choice of the conformal factor on $\tilde{S}$. This freedom is a reflection of the freedom in the choice of an origin for Newtonian multipole moments. But although the multipole moments depend on the conformal factor, the vanishing of all the multipole moments of a given type—and so, the assumption of the theorem—is independent of the particular choice of the conformal factor.

We now give the proof of the theorem.

Consider first angular momentum multipole moments. Using the first of equations (2) it is easy to show that the trace of $\tilde{D}_{a_1}P_{a_2\ldots a_{s+1}}$ is of the form $\tilde{D}_{a_1}\ldots\tilde{D}_{a_{s-1}}\tilde{\phi}_J+$ lower-order derivatives of $\tilde{\phi}_J$. Then, since also $\tilde{D}_{(a_1}\ldots\tilde{D}_{a_s)}\tilde{\phi}_J=\tilde{D}_{a_1}\ldots\tilde{D}_{a_s}\tilde{\phi}_J+$lower-order derivatives of $\tilde{\phi}_J$ one concludes that $P_{a_1\ldots a_s}=\tilde{D}_{a_1}\ldots\tilde{D}_{a_s}\tilde{\phi}_J+$lower-order derivatives of $\tilde{\phi}_J$. Hence a simple induction shows that the vanishing of all the multipole moments is equivalent to the vanishing at the point Λ of $\tilde{\phi}_J$ and all its derivatives.

Then we apply the unique continuation theorem. Choose any neighbourhood N of Λ in $\tilde{S}$ with compact closure. Since $\frac{1}{8}\tilde{R}+\frac{15}{8}\tilde{k}^4$ is smooth on $\tilde{S}$, it is bounded on N; let B_N be an upper bound for $\frac{1}{8}\tilde{R}+\frac{15}{8}\tilde{k}^4$. By virtue of the first of equations (2) the inequality (1) is immediately satisfied with linear elliptic operator $\tilde{D}^m\tilde{D}_m$ and constant B_N. So, $\tilde{\phi}_J=0$ in any such neighbourhood of Λ in $\tilde{S}$. Using a compact covering of $\tilde{S}$ we can conclude that $\tilde{\phi}_J=0$ on $\tilde{S}$. Since $\Omega\neq 0$ on S, the twist ω should vanish on S and hence the space–time is static.

For the second part of the theorem again one first shows that the vanishing of all the mass multipole moments on a static space–time implies that $\phi_M=0$ on S and hence, since $\omega=0$ too, that $\lambda=1$ in M. Moreover, equation 2.17 of Hansen (1974) (an equation on the Ricci tensor of h_{ab}, λ and ω, which is a consequence of the Einstein equation) with $\phi_M=\phi_J=0$ implies that the Ricci tensor of (S, h_{ab}) vanishes and hence $-S$ is three dimensional!—(S, h_{ab}) is flat. Since the space–time metric is $g_{ab}=h_{ab}-\xi_a\xi_b$, (M, g_{ab}) is flat too; QED.

Our theorem provides only a small portion of what is a more interesting result which is expected to be true, namely that two stationary space–times with the same sets of multipole moments are isometric at least in some neighbourhood of infinity (Hansen 1974). The theorem is not a uniqueness proof because the multipole moments do not depend linearly on the potentials $\tilde{\phi}_M$ and $\tilde{\phi}_J$, because equations (2) are not linear ($\tilde{k}^4$ depends on $\tilde{\phi}_M$, $\tilde{\phi}_J$ and their first derivatives) and because a uniqueness proof should also conclude that the corresponding three-dimensional metrics are isometric.

An interesting example is provided by the Papapetrou (1953) class of solutions. For the asymptotically flat space–times in this class *all* the mass multipole moments vanish while the angular momentum multipole moments do not. Indeed, in this class, we have for the stationary Killing field

$$\lambda=2f/(f^2+1) \qquad \text{and} \qquad \omega=(f^2-1)/(f^2+1)$$

where f is any solution of a certain second-order differential equation on a two-dimensional manifold. Since $\lambda^2+\omega^2=1$, the mass potential $\tilde{\phi}_M$ vanishes on $\tilde{S}$.

One can essentially repeat the present proof to show that a static, asymptotically flat solution of the Einstein–Maxwell equation for which all the electromagnetic multipole moments (Hoenselaers 1976) vanish is in fact a solution of the vacuum Einstein equation.

Note that the potential $\tilde{\phi}_M$ used by Hansen to define the mass multipole moments does not reduce, in the static case, to the potential used by Geroch to define multipole moments. One expects that, although the potentials are different, they eventually produce the same multipole moments, but this has been proved only for static and axisymmetric space–times† (Fette 1975, unpublished). Some more evidence that the two collections of multipole moments may be the same is provided by the theorem: one can show that a static space–time for which all the multipole moments vanish is flat by using either the Hansen or the Geroch multipole moments.

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